

Hojman Symmetry in $f(T)$ Theory

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ABSTRACT

Today, $f(T)$ theory has been one of the popular modified gravity theories to explain the accelerated expansion of the universe without invoking dark energy. In this work, we consider the so-called Hojman symmetry in $f(T)$ theory. Unlike Noether conservation theorem, the symmetry vectors and the corresponding conserved quantities in Hojman conservation theorem can be obtained by using directly the equations of motion, rather than Lagrangian or Hamiltonian. We find that Hojman symmetry can exist in $f(T)$ theory, and the corresponding exact cosmological solutions are obtained. We find that the functional form of $f(T)$ is restricted to be the power-law or hypergeometric type, while the universe experiences a power-law or hyperbolic expansion. These results are different from the ones obtained by using Noether symmetry in $f(T)$ theory. Therefore, it is reasonable to find exact cosmological solutions via Hojman symmetry.

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I. INTRODUCTION

As is well known, symmetry plays an important role in the theoretical physics. In particular, symmetry is a useful tool to select models motivated at a fundamental level, and find the exact solutions. One of the well-known symmetries in physics is Noether symmetry. In fact, Noether symmetry has been extensively used in cosmology and gravity theories, for instance, scalar field cosmology [1, 2], $f(R)$ theory [3–5, 26], scalar-tensor theory [6, 7], $f(T)$ theory [8, 25], Gauss-Bonnet gravity [9], non-minimally coupled cosmology [10], and others [11, 27, 28]. It is worth noting that a (point-like) Lagrangian should be given *a priori* when one uses Noether symmetry in these models.

In the year 1992, Hojman [12] proposed a drastically new conservation theorem constructed without using Lagrangian or Hamiltonian. Unlike Noether conservation theorem, the symmetry vectors and the corresponding conserved quantities in Hojman conservation theorem can be obtained by using directly the equations of motion, rather than Lagrangian or Hamiltonian. In general, its conserved quantities and other related quantities can be different from the ones in Noether conservation theorem.

Here is Hojman conservation theorem [12]. We consider a set of second order differential equations

$$\ddot{q}^i = F^i(q^j, \dot{q}^j, t), \quad i, j = 1, \dots, m \quad (1)$$

where a dot denotes a derivative with respect to time t . If $X^i = X^i(q^j, \dot{q}^j, t)$ is a symmetry vector for Eq. (1), it satisfies [13, 14]

$$\frac{d^2 X^i}{dt^2} - \frac{\partial F^i}{\partial \dot{q}^j} X^j - \frac{\partial F^i}{\partial \dot{q}^j} \frac{dX^j}{dt} = 0, \quad (2)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + F^i \frac{\partial}{\partial \dot{q}^i}. \quad (3)$$

The symmetry vector X^i is defined so that the infinitesimal transformation

$$\tilde{q}^i = q^i + \epsilon X^i(q^j, \dot{q}^j, t) \quad (4)$$

maps solutions q^i of Eq. (1) into solutions \tilde{q}^i of the same equations (up to ϵ^2 terms) [13, 14]. If the “force” F^i satisfies (in some coordinate systems)

$$\frac{\partial F^i}{\partial \dot{q}^i} = 0, \quad (5)$$

then

$$Q = \frac{\partial X^i}{\partial q^i} + \frac{\partial}{\partial \dot{q}^i} \left(\frac{dX^i}{dt} \right) \quad (6)$$

is a conserved quantity for Eq. (1), namely $dQ/dt = 0$. In fact, the condition (5) can be relaxed [12]. If the “force” F^i satisfies (in some coordinate systems)

$$\frac{\partial F^i}{\partial \dot{q}^i} = -\frac{d}{dt} \ln \gamma, \quad (7)$$

where $\gamma = \gamma(q^i)$ is a function of q^i , then

$$Q = \frac{1}{\gamma} \frac{\partial (\gamma X^i)}{\partial q^i} + \frac{\partial}{\partial \dot{q}^i} \left(\frac{dX^i}{dt} \right) \quad (8)$$

is a conserved quantity for Eq. (1). Obviously, if $\gamma = \text{const.}$, Eqs. (7) and (8) reduce to Eqs. (5) and (6). In the proof of this conservation theorem [12] (see also e.g. [24]), neither a Lagrangian nor a Hamiltonian is needed, and no previous knowledge of a constant of motion for system (1) is invoked either [12].

In [12], Hojman showed that this conservation theorem can drastically restrict the functional form of point symmetry transformations, and a harmonic oscillator system was considered as a concrete example. Recently, Hojman conservation theorem has been used in cosmology and gravity theory [15, 16]. It is found that Hojman conserved quantities exist for a wide range of the potential $V(\phi)$ of quintessence [15] and scalar-tensor theory [16], and the corresponding exact cosmological solutions have been obtained. As mentioned above, Hojman conserved quantities and other related quantities can be different from the ones of Noether. In fact, Noether symmetry exists only for exponential potential $V(\phi)$ [1, 6, 7], while Hojman symmetry exists for a wide range of potentials $V(\phi)$, including not only exponential but also power-law, hyperbolic, logarithmic and other complicated potentials [15, 16]. Therefore, Hojman symmetry might give rise to new features in cosmology and gravity theory.

In the present work, we are interested to consider Hojman symmetry in $f(T)$ theory. We try to restrict the functional form of $f(T)$, and find the exact cosmological solutions as well as the corresponding conserved quantities in $f(T)$ theory, by using Hojman conservation theorem. In Sec. II, we briefly review the key points of $f(T)$ theory at first. In Secs. III and IV, we then consider Hojman symmetry in $f(T)$ theory with pressureless and barotropic matter, respectively. The brief conclusion is given in Sec. V.

II. $f(T)$ THEORY

The current accelerated expansion of the universe could be due to an unknown energy component (dark energy) or a modification to general relativity (modified gravity) [17, 18]. In analogy to the well-known $f(R)$ theory, recently $f(T)$ theory [19, 20] has been proposed as a new modified gravity theory to drive the accelerated cosmic expansion without invoking dark energy. $f(T)$ theory is a generalized version of the teleparallel gravity originally proposed by Einstein [21, 22], in which the Weitzenböck connection is used, rather than the Levi-Civita connection used in general relativity. Here we briefly review the key points of $f(T)$ theory following [19, 20]. We consider a spatially flat Friedmann-Robertson-Walker (FRW) universe whose spacetime is described by

$$ds^2 = -dt^2 + a^2(t) d\mathbf{x}^2, \quad (9)$$

where a is the scale factor. The orthonormal tetrad components $e_i(x^\mu)$ relate to the metric through

$$g_{\mu\nu} = \eta_{ij} e_\mu^i e_\nu^j, \quad (10)$$

where Latin i, j are indices running over 0, 1, 2, 3 for the tangent space of the manifold, and Greek μ, ν are the coordinate indices on the manifold, also running over 0, 1, 2, 3. In $f(T)$ theory, the gravitational action is given by

$$\mathcal{S}_T = \int d^4x |e| f(T), \quad (11)$$

where $|e| = \det(e_\mu^i) = \sqrt{-g}$, and we use the units $16\pi G = \hbar = c = 1$ throughout this work. $f(T)$ is a function of the torsion scalar T , which is defined by

$$T \equiv S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu}, \quad (12)$$

with

$$T^\rho{}_{\mu\nu} \equiv -e_i^\rho (\partial_\mu e_\nu^i - \partial_\nu e_\mu^i), \quad (13)$$

$$K^{\mu\nu}{}_\rho \equiv -\frac{1}{2} (T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu}), \quad (14)$$

$$S_\rho{}^{\mu\nu} \equiv \frac{1}{2} (K^{\mu\nu}{}_\rho + \delta_\rho^\mu T^{\theta\nu}{}_\theta - \delta_\rho^\nu T^{\theta\mu}{}_\theta). \quad (15)$$

For a spatially flat FRW universe, from Eqs. (12) and (9), we find that

$$T = -6H^2, \quad (16)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter, and a dot denotes a derivative with respect to cosmic time t . In $f(T)$ theory, the modified Friedmann equation and Raychaudhuri equation read [8, 19, 20, 23]

$$12H^2 f_T + f = \rho, \quad (17)$$

$$48H^2 f_{TT} \dot{H} - f_T (12H^2 + 4\dot{H}) - f = p, \quad (18)$$

where a subscript T denotes a derivative with respect to T , while ρ, p are the total energy density and pressure, respectively. It is well known that when $f(T) = T$ the familiar general relativity can be completely recovered. Unlike metric $f(R)$ theory whose equations of motion are 4th order, the equations of motion (17) and (18) in $f(T)$ theory are 2nd order. This is one of the great virtues of $f(T)$ theory. Since Hojman symmetry mentioned above mainly deals with the equations of motion rather than Lagrangian or Hamiltonian, this virtue makes it easy.

III. HOJMAN SYMMETRY IN $f(T)$ THEORY WITH PRESSURELESS MATTER

Here we study Hojman symmetry in $f(T)$ theory. To be simple, we first consider a flat FRW universe containing only pressureless matter, namely the pressure $p = p_m = 0$ and the energy density $\rho = \rho_m = \rho_{m0} a^{-3}$, where the subscript “0” indicates the present value of the corresponding quantity, and we have set $a_0 = 1$. Using Eqs. (18) and (16), we have

$$\dot{H} = \frac{12H^2 f_T + f}{48H^2 f_{TT} - 4f_T}. \quad (19)$$

Following [15, 16], we introduce a new variable $x \equiv \ln a$, and recast Eq. (19) as

$$\ddot{x} = -\frac{1}{4} \cdot \frac{f - 2T f_T}{2T f_{TT} + f_T} = F(\dot{x}). \quad (20)$$

Noting that f and its derivatives with respect to T are all functions of $T = -6H^2$, the “force” F explicitly depends only on $\dot{x} = H$. If Hojman symmetry exists in $f(T)$ theory, the condition (7) should be satisfied. Noting Eq. (3) and $\gamma = \gamma(x)$, we recast Eq. (7) as

$$-\frac{1}{\dot{x}} \frac{\partial F(\dot{x})}{\partial \dot{x}} = \frac{\partial}{\partial x} \ln \gamma(x). \quad (21)$$

Since its left-hand side is a function of \dot{x} only and its right-hand side is a function of x only, they must be equal to a same constant in order to ensure that Eq. (21) always holds. For convenience, we let this constant be $3/n$ while $n \neq 0$, and then Eq. (21) can be separated into two ordinary differential equations

$$\frac{\partial}{\partial x} \ln \gamma(x) = \frac{3}{n}, \quad \frac{\partial F(\dot{x})}{\partial \dot{x}} = -\frac{3}{n} \dot{x}. \quad (22)$$

Thus, we find that

$$\gamma(x) = \gamma_0 e^{3x/n}, \quad (23)$$

$$F(\dot{x}) = -\frac{3}{2n} \dot{x}^2 + c_0, \quad (24)$$

where γ_0 and c_0 are both integral constants. In the following subsections, we consider the cases of $c_0 = 0$ and $c_0 \neq 0$, respectively.

A. The case of $c_0 = 0$

In the case of $c_0 = 0$, substituting $F = -3\dot{x}^2/(2n) = T/(4n)$ into Eq. (20), we have

$$2T^2 f_{TT} + (1 - 2n) T f_T + n f = 0, \quad (25)$$

which is a differential equation of $f(T)$ with respect to T . Its solution is given by

$$f(T) = c_1 T^n + c_2 \sqrt{T}, \quad (26)$$

where c_1 and c_2 are both integral constants. On the other hand, from $\dot{H} = \ddot{x} = F = -3H^2/(2n)$, we get

$$H(t) = \frac{2n}{3}(t + c_3)^{-1}, \quad (27)$$

where c_3 is an integral constant. Noting that $H = \dot{a}/a$, it is easy to obtain

$$a(t) = c_4(t + c_3)^{2n/3}, \quad (28)$$

where c_4 is an integral constant. One might set $c_3 = 0$ by requiring $a(t = 0) = 0$. Note that $n > 0$ is required to ensure that the universe is expanding. Using Eqs. (27) and (28), it is easy to find the dimensionless Hubble parameter

$$E \equiv H/H_0 = a^{-3/(2n)} = (1 + z)^{3/(2n)}, \quad (29)$$

where z is the redshift. Using $\dot{H} = \ddot{x} = F = -3H^2/(2n)$, we obtain the deceleration parameter

$$q \equiv -\frac{\ddot{a}}{aH^2} = -1 - \frac{\dot{H}}{H^2} = \frac{3}{2n} - 1. \quad (30)$$

When $n > 3/2$, the expansion of the universe can be accelerated (note that $n < 0$ is not acceptable because the universe contracts in this case).

Let us turn to the conserved quantity. Following [15, 16], we assume that the symmetry vector X does not explicitly depend on time. Substituting Eq. (24) with $c_0 = 0$ into Eq. (2), the equation for X reads

$$\frac{\partial^2 X}{\partial x^2} - \frac{3\dot{x}}{n} \frac{\partial^2 X}{\partial x \partial \dot{x}} + \frac{9\dot{x}^2}{4n^2} \frac{\partial^2 X}{\partial \dot{x}^2} + \frac{3}{2n} \frac{\partial X}{\partial x} = 0. \quad (31)$$

To solve this equation, we adopt the ansatz

$$X = A_0 \dot{x}^\alpha e^{\beta x} + A_1, \quad (32)$$

where A_0, A_1, α, β are all constants, and α, β cannot be zero at the same time. Substituting Eq. (32) into Eq. (31), we find that the solutions have $3\alpha - 2n\beta = 0$ or $3\alpha - 2n\beta = 3$. Substituting Eqs. (32) and (23) into Eq. (8), the conserved quantity Q is given by

$$Q = \frac{1}{2n} [6A_1 - (2 + \alpha)(3\alpha - 2n\beta - 3)A_0 \dot{x}^\alpha e^{\beta x}]. \quad (33)$$

If $3\alpha - 2n\beta = 3$ or $\alpha = -2$, then $Q = 3A_1/n = \text{const.}$ is trivial. If $3\alpha - 2n\beta = 0$ and $\alpha \neq -2$, we get

$$\dot{x}^{2n} e^{3x} = \text{const.} \quad (34)$$

In fact, this conserved quantity can be found in another way. Substituting Eq. (26) and $T = -6H^2 = -6\dot{x}^2$ as well as $\rho = \rho_m = \rho_{m0} a^{-3} = \rho_{m0} e^{-3x}$ into Eq. (17), one can find the same conserved quantity given in Eq. (34) again. This can be regarded as a confirmation of Hojman conservation theorem.

B. The case of $c_0 \neq 0$

In the case of $c_0 \neq 0$, substituting Eq. (24) into Eq. (20), and noting $-3\dot{x}^2 = T/2$, we obtain a differential equation of $f(T)$ with respect to T ,

$$-\frac{1}{4} \cdot \frac{f - 2Tf_T}{2Tf_{TT} + f_T} = \frac{T}{4n} + c_0. \quad (35)$$

The corresponding solution for $c_0 \neq 0$ is given by

$$f(T) = c_1(4nc_0)^n \cdot {}_2F_1\left(-\frac{1}{2}, -n; \frac{1}{2}; -\frac{T}{4nc_0}\right) + c_2\sqrt{T}, \quad (36)$$

where ${}_2F_1$ is a hypergeometric function, and c_1, c_2 are both integral constants. In the limit $c_0 \rightarrow 0$, Eq. (36) can reduce to Eq. (26). Obviously, in general $c_0 \neq 0$ makes difference. From $\dot{H} = \ddot{x} = F = -3\dot{x}^2/(2n) + c_0 = -3H^2/(2n) + c_0$, we obtain

$$H(t) = \sqrt{\frac{2nc_0}{3}} \tanh\left(\sqrt{\frac{3c_0}{2n}}(t + c_3)\right), \quad (37)$$

and then

$$a(t) = c_4 \left[\cosh\left(\sqrt{\frac{3c_0}{2n}}(t + c_3)\right) \right]^{2n/3}, \quad (38)$$

where c_3 and c_4 are both integral constants. Using Eqs. (37) and (38), it is easy to obtain the Hubble parameter as a function of scale factor

$$H(a) = \sqrt{\frac{2nc_0}{3}} \tanh\left(\operatorname{arccosh}\left(\left(\frac{a}{c_4}\right)^{\frac{3}{2n}}\right)\right), \quad (39)$$

and then $E = H/H_0$ is ready. On the other hand, using $\dot{H} = \ddot{x} = F = -3\dot{x}^2/(2n) + c_0 = -3H^2/(2n) + c_0$, we also obtain the deceleration parameter

$$q \equiv -\frac{\ddot{a}}{aH^2} = -1 - \frac{\dot{H}}{H^2} = \frac{3}{2n} - 1 - \frac{c_0}{H^2}, \quad (40)$$

where H is given in Eq. (37) or (39).

Let us turn to the conserved quantity. Substituting Eq. (24) with $c_0 \neq 0$ into Eq. (2), we get the equation for the symmetry vector X . Some terms with $c_0 \neq 0$ appear, comparing with Eq. (31). The ansatz in Eq. (32) does not work, mainly due to the additional term $\sim c_0^2 \partial^2 X / \partial \dot{x}^2$ (because the order of \dot{x} in this term is different from other terms). So, it is hard to solve Eq. (2) with $c_0 \neq 0$ to get the symmetry vector X . Thus, the task to obtain the conserved quantity Q in Eq. (8) is also hard. Fortunately, there exists another way. Inspired by the discussion below Eq. (34), we can instead find the corresponding conserved quantity by using the modified Friedmann equation (17). Substituting Eq. (36) and $T = -6H^2 = -6\dot{x}^2$ as well as $\rho = \rho_m = \rho_{m0} a^{-3} = \rho_{m0} e^{-3x}$ into Eq. (17), it is easy to find the conserved quantity

$$\left(\dot{x}^2 - \frac{2nc_0}{3}\right)^n e^{3x} = \text{const.} \quad (41)$$

Obviously, it reduces to Eq. (34) if $c_0 = 0$. In fact, Eq. (41) suggests us to instead adopt another ansatz $X = A_0(\dot{x}^2 + A_3)^\alpha e^{\beta x} + A_1$ (or something similar) to solve the equation for X , and then obtain the conserved quantity Q in Eq. (8) again. Although this is anticipated to work well but the result must be the same one given in Eq. (41), here we do not try the complicated calculation. Let us move forward.

IV. HOJMAN SYMMETRY IN $f(T)$ THEORY WITH BAROTROPIC MATTER

In this section, we consider a more general case, and assume a flat FRW universe containing the so-called barotropic matter, i.e. the pressure $p = p_m = w\rho_m$ and the energy density $\rho = \rho_m = \rho_{m0} a^{-3(1+w)}$, where the constant $w \neq -1$ is the equation-of-state parameter (EoS) of barotropic matter. Using Eqs. (18), (17) and (16) with $p = w\rho$, we have

$$\dot{H} = \frac{(1+w)(12H^2 f_T + f)}{48H^2 f_{TT} - 4f_T}, \quad (42)$$

which can be recast as

$$\ddot{x} = -\frac{1+w}{4} \cdot \frac{f - 2Tf_T}{2Tf_{TT} + f_T} = F(\dot{x}). \quad (43)$$

Noting that f and its derivatives with respect to T are all functions of $T = -6H^2$, the “force” F explicitly depends only on $\dot{x} = H$. If Hojman symmetry exists in $f(T)$ theory, the condition (7) should be satisfied. Noting Eq. (3) and $\gamma = \gamma(x)$, we recast Eq. (7) to the one given in Eq. (21). Following the same derivation between Eqs. (21)–(24), we obtain the same $\gamma(x)$ and $F(\dot{x})$ given in Eqs. (23) and (24). In the following subsections, we consider the cases of $c_0 = 0$ and $c_0 \neq 0$, respectively.

A. The case of $c_0 = 0$

In the case of $c_0 = 0$, substituting $F = -3\dot{x}^2/(2n) = T/(4n)$ into Eq. (43), we have

$$2T^2 f_{TT} + (1 - 2\tilde{n}) T f_T + \tilde{n} f = 0, \quad (44)$$

where $\tilde{n} = n(1+w)$. This is a differential equation of $f(T)$ with respect to T . Its solution is given by

$$f(T) = c_1 T^{\tilde{n}} + c_2 \sqrt{T} = c_1 T^{n(1+w)} + c_2 \sqrt{T}, \quad (45)$$

where c_1 and c_2 are both integral constants. On the other hand, from $\dot{H} = \ddot{x} = F = -3H^2/(2n)$, we find the same $H(t)$, $a(t)$, $E \equiv H/H_0$ and q given in Eqs. (27)–(30), which have not been affected by the EoS of barotropic matter w .

Then we turn to the conserved quantity. Following [15, 16], we assume that the symmetry vector X does not explicitly depend on time. Substituting Eq. (24) with $c_0 = 0$ into Eq. (2), the equation for X becomes the same one given in Eq. (31). Following the same derivation between Eqs. (31)–(34), we obtain the same conserved quantity given in Eq. (34). At first glance, this result is surprising, because it also has not been affected by the EoS of barotropic matter w . Let us check it in another way. Substituting Eq. (45) and $T = -6H^2 = -6\dot{x}^2$ as well as $\rho = \rho_m = \rho_{m0} a^{-3(1+w)} = \rho_{m0} e^{-3x(1+w)}$ into Eq. (17), one find the same conserved quantity given in Eq. (34) once again, since the factor $1+w = \text{const.}$ (note that $w \neq -1$) in both ρ and $f(T)$ can be removed at the same time. Therefore, it is not so surprising that the conserved quantity has the same form given in Eq. (34).

B. The case of $c_0 \neq 0$

In the case of $c_0 \neq 0$, substituting Eq. (24) into Eq. (43), and noting $-3\dot{x}^2 = T/2$, we obtain a differential equation of $f(T)$ with respect to T ,

$$-\frac{1}{4} \cdot \frac{f - 2Tf_T}{2Tf_{TT} + f_T} = \frac{T}{4\tilde{n}} + \tilde{c}_0, \quad (46)$$

where $\tilde{n} = n(1+w)$ and $\tilde{c}_0 = c_0/(1+w)$. Note that $w \neq -1$. The solution for $c_0 \neq 0$ is given by

$$\begin{aligned} f(T) &= c_1 (4\tilde{n}\tilde{c}_0)^{\tilde{n}} \cdot {}_2F_1 \left(-\frac{1}{2}, -\tilde{n}; \frac{1}{2}; -\frac{T}{4\tilde{n}\tilde{c}_0} \right) + c_2 \sqrt{T} \\ &= c_1 (4nc_0)^{n(1+w)} \cdot {}_2F_1 \left(-\frac{1}{2}, -n(1+w); \frac{1}{2}; -\frac{T}{4nc_0} \right) + c_2 \sqrt{T}, \end{aligned} \quad (47)$$

where ${}_2F_1$ is a hypergeometric function, and c_1, c_2 are both integral constants. On the other hand, from $\dot{H} = \ddot{x} = F = -3\dot{x}^2/(2n) + c_0 = -3H^2/(2n) + c_0$, we obtain the same $H(t)$, $a(t)$, $H(a)$ and q given in Eqs. (37)–(40), which have not been affected by the EoS of barotropic matter w .

Let us turn to the conserved quantity. Similar to the discussions in the second part of Sec. IIIB, it is hard to solve the equation for the symmetry vector X , mainly due to the additional term $\sim c_0^2 \partial^2 X / \partial \dot{x}^2$.

Thus, the task to obtain the conserved quantity Q in Eq. (8) is also hard. Fortunately, there exists another way. Inspired by the discussion below Eq. (34), we can instead find the corresponding conserved quantity by using the modified Friedmann equation (17). Substituting Eq. (47) and $T = -6H^2 = -6\dot{x}^2$ as well as $\rho = \rho_m = \rho_{m0} a^{-3(1+w)} = \rho_{m0} e^{-3x(1+w)}$ into Eq. (17), it is easy to find that the conserved quantity has the same form given in Eq. (41). Again, it also has not been affected by the EoS of barotropic matter w .

V. CONCLUSION

Today, $f(T)$ theory has been one of the popular modified gravity theories to explain the accelerated expansion of the universe without invoking dark energy. In this work, we consider the so-called Hojman symmetry in $f(T)$ theory. Unlike Noether conservation theorem, the symmetry vectors and the corresponding conserved quantities in Hojman conservation theorem can be obtained by using directly the equations of motion, rather than Lagrangian or Hamiltonian. We consider $f(T)$ theory with pressureless and barotropic matter, respectively. We find that Hojman symmetry can exist in $f(T)$ theory, and the corresponding exact cosmological solutions are obtained. The integral constant c_0 in the “force” F plays an important role. If $c_0 = 0$, the corresponding functional form of $f(T)$ is restricted to be the power-law type, while the universe experiences a power-law expansion. The EoS of barotropic matter w does not change these key features. It is worth noting that the same results were found by using Noether symmetry in $f(T)$ theory [8]. However, if $c_0 \neq 0$, the results are drastically changed. In the case of $c_0 \neq 0$, the corresponding functional form of $f(T)$ is restricted to be the hypergeometric type, while the universe experiences a hyperbolic expansion. The EoS of barotropic matter w does not change these key features. Note that these new results cannot be found by using Noether symmetry in $f(T)$ theory.

As is well known, usually a symmetry leads to a conserved quantity. In this work, we also explicitly derived the corresponding conserved quantities for Hojman symmetry in $f(T)$ theory. They are given in Eqs. (34) and (41) for the cases of $c_0 = 0$ and $c_0 \neq 0$, respectively. Again, the EoS of barotropic matter w does not change these key features. Usually, the conserved quantities are useful and helpful in many complicated cases, since one needs not to deal with the middle evolution processes.

In some sense, Hojman symmetry is wider than Noether symmetry, and our work confirms this point previously found in [15, 16] which considered Hojman symmetry in quintessence and scalar-tensor theory. So, it is reasonable to find exact cosmological solutions via Hojman symmetry. In fact, it might open a new window in the field of cosmology and gravity theory, and bring new features to them.

Obviously, it is very natural to further consider Hojman symmetry in other cosmological models and gravity theories. In fact, we found that it is hard to use Hojman symmetry in metric $f(R)$ theory, since its equations of motion are 4th order. However, the corresponding equations of motion in Palatini $f(R)$ theory are 2nd order [18] (we thank the referee for pointing out this issue), similar to the case of $f(T)$ theory. So, it is of interest to consider Hojman symmetry in $f(R)$ theory in the Palatini formalism. We leave it to the future works.

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